Exercises on Graph Isomorphism
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Two undirected, simple graphs G and H are *isomorphic*, denoted $G \cong H$, if there is a bijection $\pi\colon V \to U$ that maps edges to edges and non-edges to non-edges, that is, for all $x, y \in V(G)$,

$$(u, v) \in E(G) \iff (\pi(u), \pi(v)) \in E(H).$$

Such a π is called an *isomorphism* from G to H. The Graph Isomorphism Problem is:

> Graph Isomorphism (GI)
> *Input:* Two graphs G, H
> *Decide:* Is $G \cong H$?

1. Show that GI is in NP. Is it in coNP?

2. A *vertex-colored graph* is a pair (G, c) consisting of a graph G and a function $c\colon G \to C$, assigning a "color" (element of the set C) to each vertex of G. Two vertex-colored graphs are isomorphic if there is an ordinary graph isomorphism that also preserves colors, that is, $c(v) = c(\pi(v))$ for all $v \in V(G)$. Show that Vertex-Colored GI is Karp equivalent (i.e. $\equiv_m^p$, i.e. polynomial-time many-one) to ordinary GI. Such problems are sometimes called GI-complete.

3. (Search to decision reduction) Search GI is the problem to output an isomorphism between two graphs if one exists, or report that they are not isomorphic. Show that Search GI reduces to GI (is in P^{GI}). *Hint:* Similar to the search-to-decision reduction for SAT, how can you figure out just one part of a witness (an isomorphism)?

Resources

- The current record in terms of theoretical upper bounds on GI is Babai's quasi-polynomial-time algorithm (exposition in Helfgott, Bajpai, and Dona. The group-theoretic method that it is based on, going back to Babai '79 "Monte-Carlo algorithms in graph isomorphism testing" and then developed using more in-depth group theory by Luks in 1980, is explained in the book by Hoffman and Paolo Codenotti's thesis.

- Existing software for GI is very effective (so much so that in practice, the problem is solved). Check out nauty, Traces, saucy, bliss, conauto. Some of these are discussed in McKay & Piperno.

- Köbler, Schöning, & Torán is the only book I know of to focus on the structural complexity of GI. It's quite good, and covers many complexity topics from this course!